Localization formulas about two Killing vector fields

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Abstract

In this article, we will discuss the smooth $(X_M+\sqrt{-1}Y_M)$-invariant forms on M and to establish a localization formulas. As an application, we get a localization formulas for characteristic numbers.

The localization theorem for equivariant differential forms was obtained by Berline and Vergne(see [2]). They discuss on the zero points of a Killing vector field. Now, We will discuss on the points about two Killing vector fields and to establish a localization formulas.

Let M be a smooth closed oriented manifold. Let G be a compact Lie group acting smoothly on M, and let $\mathfrak{g}$ be its Lie algebra. Let g^{TM} be a G-invariant metric on TM. If $X, Y \in \mathfrak{g}$, let X_M, Y_M be the corresponding smooth vector field on M. If $X, Y \in \mathfrak{g}$, then X_M, Y_M are Killing vector field. Here we will introduce the equlvariant cohomology by two Killing vector fields.

1 Equlvariant cohomology by two Killing vector fields

First, let us review the definition of equlvariant cohomology by a Killing vector field. Let $\Omega^*(M)$ be the space of smooth differetial forms on M, the de Rham complex is $(\Omega^*(M), d)$. Let L_{X_M} be the Lie derivative of X_M on $\Omega^*(M)$, i_{X_M} be the interior multiplication induced by the contraction of X_M.

Set

$$d_X = d + i_{X_M},$$

then $d_X^2 = L_{X_M}$ by the following Cartan formula

$$L_{X_M} = [d, i_{X_M}].$$

Let

$$\Omega_X^*(M) = \{\omega \in \Omega^*(M) : L_{X_M}\omega = 0\}$$

be the space of smooth X_M-invariant forms on M. Then $d_X^2\omega = 0$, when $\omega \in \Omega_X^*(M)$. It is a complex $(\Omega_X^*(M), d_X)$. The corresponding cohomology group

$$H_X^*(M) = \frac{\mathrm{Kerd_X}|_{\Omega_X^*(M)}}{\mathrm{Imd_X}|_{\Omega_X^*(M)}}$$

is called the equivariant cohomology associated with X. If a form ω has $d_X\omega = 0$, then ω called d_X-closed form.

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Then we will to definite a new complex by two Killing vector field. If $X, Y \in \mathfrak{g}$, let X_M, Y_M be the corresponding smooth vector field on M.

We know

$$L_{X_M} + \sqrt{-1}L_{Y_M}$$

be the operator on $\Omega^*(M) \otimes_{\mathbb{R}} \mathbb{C}$.

Set

$$i_{X_M+\sqrt{-1}Y_M} \doteq i_{X_M} + \sqrt{-1}i_{Y_M}$$

be the interior multiplication induced by the contraction of $X_M + \sqrt{-1}Y_M$. It is also a operator on $\Omega^*(M) \otimes_{\mathbb{R}} \mathbb{C}$.

Set

$$d_{X+\sqrt{-1}Y} = d + i_{X_M+\sqrt{-1}Y_M}.$$

Lemma 1. *If $X, Y \in \mathfrak{g}$, let X_M, Y_M be the corresponding smooth vector field on M; then*

$$d^2_{X+\sqrt{-1}Y} = L_{X_M} + \sqrt{-1}L_{Y_M}$$

Proof.

$$\begin{aligned}(d + i_{X_M+\sqrt{-1}Y_M})^2 &= (d + i_{X_M} + \sqrt{-1}i_{Y_M})(d + i_{X_M} + \sqrt{-1}i_{Y_M}) \\ &= d^2 + di_{X_M} + i_{X_M}d + \sqrt{-1}di_{Y_M} + \sqrt{-1}i_{Y_M}d + (i_{X_M} + \sqrt{-1}i_{Y_M})^2 \\ &= L_{X_M} + \sqrt{-1}L_{Y_M}\end{aligned}$$

□

Let

$$\Omega^*_{X_M+\sqrt{-1}Y_M}(M) = \{\omega \in \Omega^*(M) \otimes_{\mathbb{R}} \mathbb{C} : (L_{X_M} + \sqrt{-1}L_{Y_M})\omega = 0\}$$

be the space of smooth $(X_M + \sqrt{-1}Y_M)$-invariant forms on M. Then we get a complex $(\Omega^*_{X_M+\sqrt{-1}Y_M}(M), d_{X+\sqrt{-1}Y})$. We call a form ω is $d_{X+\sqrt{-1}Y}$-closed if $d_{X+\sqrt{-1}Y}\omega = 0$ (this is first discussed by Bimsut, see [3]).The corresponding cohomology group

$$H^*_{X+\sqrt{-1}Y}(M) = \frac{\mathrm{Kerd}_{\mathrm{X+\sqrt{-1}Y}}|_{\Omega^*_{\mathrm{X+\sqrt{-1}Y}}(\mathrm{M})}}{\mathrm{Imd}_{\mathrm{X+\sqrt{-1}Y}}|_{\Omega^*_{\mathrm{X+\sqrt{-1}Y}}(\mathrm{M})}}$$

is called the equivariant cohomology associated with K.

2 The set of zero points

Lemma 2. *If $X, Y \in \mathfrak{g}$, let X_M, Y_M be the corresponding smooth vector field on M, $X^{'}, Y^{'}$ be the 1-form on M which is dual to X_M, Y_M by the metric g^{TM}, then*

$$L_{X_M}Y^{'} + L_{Y_M}X^{'} = 0$$

Proof. Because

$$(L_{X_M}\omega)(Z) = X_M(\omega(Z)) - \omega([X_M, Z])$$

here $Z \in \Gamma(TM)$, So we get

$$(L_{X_M}Y^{'})(Z) = X_M < Y_M, Z > - < [X_M, Z], Y_M >$$

$$(L_{Y_M}X^{'})(Z) = Y_M < X_M, Z > - < [Y_M, Z], X_M > .$$

Because X_M, Y_M are Killing vector fields, so (see [6])

$$\begin{aligned} X_M < Y_M, Z > &=< L_{X_M}Y_M, Z > + < Y_M, L_{X_M}Z > \\ &=< [X_M, Y_M], Z > + < Y_M, [X_M, Z] > \end{aligned}$$

$$\begin{aligned} Y_M < X_M, Z > &=< L_{Y_M}X_M, Z > + < X_M, L_{Y_M}Z > \\ &=< [Y_M, X_M], Z > + < X_M, [Y_M, Z] > \end{aligned}$$

then we get

$$(L_{X_M}Y^{'} + L_{Y_M}X^{'})(Z) =< [X_M, Y_M], Z > + < [Y_M, X_M], Z >= 0$$

□

Lemma 3. *If $X, Y \in \mathfrak{g}$, let X_M, Y_M be the corresponding smooth vector field on M, $X^{'}, Y^{'}$ be the 1-form on M which is dual to X_M, Y_M by the metric g^{TM}, then*

$$d_{X+\sqrt{-1}Y}(X^{'} + \sqrt{-1}Y^{'})$$

is the $d_{X+\sqrt{-1}Y}$-closed form.

Proof.

$$\begin{aligned} d^2_{X+\sqrt{-1}Y}(X^{'} + \sqrt{-1}Y^{'}) &= d_{X+\sqrt{-1}Y}(d(X^{'} + \sqrt{-1}Y^{'}) + i_{X_M+\sqrt{-1}Y_M}(X^{'} + \sqrt{-1}Y^{'})) \\ &= di_{X_M+\sqrt{-1}Y_M}(X^{'} + \sqrt{-1}Y^{'}) + i_{X_M+\sqrt{-1}Y_M}d(X^{'} + \sqrt{-1}Y^{'}) \\ &= L_{X_M}X^{'} - L_{Y_M}Y^{'} + \sqrt{-1}(L_{X_M}Y^{'} + L_{Y_M}X^{'}) \\ &= 0 \end{aligned}$$

So $d_{X+\sqrt{-1}Y}(X^{'} + \sqrt{-1}Y^{'})$ is the $d_{X+\sqrt{-1}Y}$-closed form. □

Lemma 4. *For any $\eta \in H^*_{X+\sqrt{-1}Y}(M)$ and $s \geq 0$, we have*

$$\int_M \eta = \int_M \exp\{-s(d_{X+\sqrt{-1}Y}(X^{'} + \sqrt{-1}Y^{'}))\}\eta$$

Proof. Because

$$\begin{aligned} &\frac{\partial}{\partial s}\int_M \exp\{-s(d_{X+\sqrt{-1}Y}(X^{'} + \sqrt{-1}Y^{'}))\}\eta \\ &= -\int_M (d_{X+\sqrt{-1}Y}(X^{'} + \sqrt{-1}Y^{'}))\exp\{-s(d_{X+\sqrt{-1}Y}(X^{'} + \sqrt{-1}Y^{'}))\}\eta \end{aligned}$$

and by assumption we have

$$d_{X+\sqrt{-1}Y}\eta = 0$$

$$d_{X+\sqrt{-1}Y}\exp\{-s(d_{X+\sqrt{-1}Y}(X^{'} + \sqrt{-1}Y^{'}))\} = 0$$

So we get

$$\begin{aligned} &(d_{X+\sqrt{-1}Y}(X^{'} + \sqrt{-1}Y^{'}))\exp\{-s(d_{X+\sqrt{-1}Y}(X^{'} + \sqrt{-1}Y^{'}))\}\eta \\ &= d_{X+\sqrt{-1}Y}[(X^{'} + \sqrt{-1}Y^{'})\exp\{-s(d_{X+\sqrt{-1}Y}(X^{'} + \sqrt{-1}Y^{'}))\}\eta] \end{aligned}$$

and by Stokes formula we have

$$\frac{\partial}{\partial s}\int_M \exp\{-s(d_{X+\sqrt{-1}Y}(X^{'}+\sqrt{-1}Y^{'}))\}\eta = 0$$

Then we get

$$\int_M \eta = \int_M \exp\{-s(d_{X+\sqrt{-1}Y}(X^{'}+\sqrt{-1}Y^{'}))\}\eta$$

□

We have

$$d_{X+\sqrt{-1}Y}(X^{'}+\sqrt{-1}Y^{'}) = d(X^{'}+\sqrt{-1}Y^{'}) + \langle X_M+\sqrt{-1}Y_M, X_M+\sqrt{-1}Y_M\rangle$$

and

$$\langle X_M+\sqrt{-1}Y_M, X_M+\sqrt{-1}Y_M\rangle = |X_M|^2 - |Y_M|^2 + 2\sqrt{-1}\langle X_M, Y_M\rangle$$

Set

$$M_0 = \{x \in M \mid \langle X_M(x)+\sqrt{-1}Y_M(x), X_M(x)+\sqrt{-1}Y_M(x)\rangle = 0\}.$$

For simplicity, we assume that M_0 is the connected submanifold of M, and $\mathcal{N}$ is the normal bundle of M_0 about M. The set M_0 is first discussed by H.Jacobowitz (see [4]).

3 Localization formula on $d_{X+\sqrt{-1}Y}$-closed form

Set E is a G-equivariant vector bundle, if ∇^E is a connection on E which commutes with the action of G on $\Omega(M,E)$, we see that

$$[\nabla^E, L_X^E] = 0$$

for all $X \in \mathfrak{g}$. Then we can get a moment map by

$$\mu^E(X) = L_X^E - [\nabla^E, i_X] = L_X^E - \nabla_X^E$$

We known that if y be the tautological section of the bundle π^*E over E, then the vertical component of X_E may be identified with $-\mu^E(X)y$(see [1] proposition 7.6).

If E is the tangent bundle TM and ∇^{TM} is Levi-Civita connection, then we have

$$\mu^{TM}(X)Y = L_XY - \nabla_X^{TM}Y = -\nabla_Y^{TM}X$$

We known that for any Killing vector field X, $\mu^{TM}(X)$ as linear endomorphisms of TM is skew-symmetric, $-\mu^{TM}(X)$ annihilates the tangent bundle TM_0 and induces a skew-symmetric automorphism of the normal bundle $\mathcal{N}$(see [5] chapter II, proposition 2.2 and theorem 5.3). The restriction of $\mu^{TM}(X)$ to $\mathcal{N}$ coincides with the moment endomorphism $\mu^{\mathcal{N}}(X)$.

Let G_0 be the Lie subgroup of G which preserves the submanifold M_0, e.g. Let $p \in M_0$, $Z \in \mathfrak{g}_o$, we have $exp(-tZ)p = q \in M_0$, here $\mathfrak{g}_o$ is the Lie algebra of G_0. We assume that the local 1-parameter transformations $\exp(-tX), \exp(-tY) \in G_0$. We have that G_0 acts on the normal bundle $\mathcal{N}$. The vector field $X^{\mathcal{N}}$ and $Y^{\mathcal{N}}$ are vertical and are given at the point $(x,y) \in M_0 \times \mathcal{N}_x$ by the vectors $-\mu^{\mathcal{N}}(X)y, -\mu^{\mathcal{N}}(Y)y \in \mathcal{N}_x$.

We construct a one-form α on $\mathcal{N}$:

$$Z \in \Gamma(T\mathcal{N}) \to \alpha(Z) =< -\mu^{\mathcal{N}}(X)y, \nabla_Z^{\mathcal{N}}y > +\sqrt{-1} < -\mu^{\mathcal{N}}(Y)y, \nabla_Z^{\mathcal{N}}y >$$

Let $Z_1, Z_2 \in \Gamma(T\mathcal{N})$, we known $d\alpha(Z_1, Z_2) = Z_1\alpha(Z_2) - Z_2\alpha(Z_1) - \alpha([Z_1, Z_2])$, so

$$\begin{aligned} d\alpha(Z_1, Z_2) =& < -\nabla^{\mathcal{N}}_{Z_1}\mu^{\mathcal{N}}(X)y, \nabla^{\mathcal{N}}_{Z_2}y > - < -\nabla^{\mathcal{N}}_{Z_2}\mu^{\mathcal{N}}(X)y, \nabla^{\mathcal{N}}_{Z_1}y > \\ &+ \sqrt{-1} < -\nabla^{\mathcal{N}}_{Z_1}\mu^{\mathcal{N}}(Y)y, \nabla^{\mathcal{N}}_{Z_2}y > -\sqrt{-1} < -\nabla^{\mathcal{N}}_{Z_2}\mu^{\mathcal{N}}(Y)y, \nabla^{\mathcal{N}}_{Z_1}y > \\ &+ < -\mu^{\mathcal{N}}(X)y, R^{\mathcal{N}}(Z_1, Z_2)y > +\sqrt{-1} < -\mu^{\mathcal{N}}(Y)y, R^{\mathcal{N}}(Z_1, Z_2)y > \end{aligned}$$

Recall that $\nabla^{\mathcal{N}}$ is invariant under L_X for all $X \in \mathfrak{g}$, so that $[\nabla^{\mathcal{N}}, \mu^{\mathcal{N}}(X)] = 0$, $[\nabla^{\mathcal{N}}, \mu^{\mathcal{N}}(Y)] = 0$. And by X, Y are Killing vector field, we have $d\alpha$ equals

$$2 < -(\mu^{\mathcal{N}}(X) + \sqrt{-1}\mu^{\mathcal{N}}(Y))\cdot, \cdot > + < -\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y, R^{\mathcal{N}}y >$$

And by $|X_{\mathcal{N}}|^2 =< \mu^{\mathcal{N}}(X)y, \mu^{\mathcal{N}}(X)y >$, $|Y_{\mathcal{N}}|^2 =< \mu^{\mathcal{N}}(Y)y, \mu^{\mathcal{N}}(Y)y >$. So We can get

$$\begin{aligned} d_{X_{\mathcal{N}}+\sqrt{-1}Y_{\mathcal{N}}}(X'_{\mathcal{N}} + \sqrt{-1}Y'_{\mathcal{N}}) =& -2 < (\mu^{\mathcal{N}}(X) + \sqrt{-1}\mu^{\mathcal{N}}(Y))\cdot, \cdot > \\ &+ < -\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y, -\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y + R^{\mathcal{N}}y > \end{aligned}$$

Theorem 1. *Let M be a smooth closed oriented manifold, G be a compact Lie group acting smoothly on M. For any $\eta \in H^*_{X+\sqrt{-1}Y}(M)$, $[X_M, Y_M] = 0$, let G_0 be the Lie subgroup of G which preserves the submanifold M_0 and the local 1-parameter transformations* $\exp(-tX), \exp(-tY) \in G_0$, *the following identity hold:*

$$\int_M \eta = \int_{M_0} \frac{\eta}{\mathrm{Pf}[\frac{-\mu^{\mathcal{N}}(\mathrm{X}) - \sqrt{-1}\mu^{\mathcal{N}}(\mathrm{Y}) + \mathrm{R}^{\mathcal{N}}}{2\pi}]}$$

Proof. Set $s = \frac{1}{2t}$, so by Lemma 4. we get

$$\int_M \eta = \int_M \exp\{-\frac{1}{2t}(d_{X+\sqrt{-1}Y}(X' + \sqrt{-1}Y'))\}\eta$$

Let V is a neighborhood of M_0 in $\mathcal{N}$. We identify a tubular neighborhood of M_0 in M with V. Set $V' \subset V$. When $t \to 0$, because $\langle X_M(x) + \sqrt{-1}Y_M(x), X_M(x) + \sqrt{-1}Y_M(x)\rangle \neq 0$ out of M_0, so we have

$$\int_M \exp\{-\frac{1}{2t}(d_{X+\sqrt{-1}Y}(X' + \sqrt{-1}Y'))\}\eta \sim \int_{V'} \exp\{-\frac{1}{2t}(d_{X+\sqrt{-1}Y}(X' + \sqrt{-1}Y'))\}\eta.$$

Because

$$\int_{V'} \exp\{-\frac{1}{2t}(d_{X+\sqrt{-1}Y}(X' + \sqrt{-1}Y'))\}\eta = \int_{V'} \exp\{-\frac{1}{2t}(d_{X_{\mathcal{N}}+\sqrt{-1}Y_{\mathcal{N}}}(X'_{\mathcal{N}} + \sqrt{-1}Y'_{\mathcal{N}}))\}\eta$$

then

$$\begin{aligned} &\int_{V'} \exp\{-\frac{1}{2t}(d_{X+\sqrt{-1}Y}(X' + \sqrt{-1}Y'))\}\eta = \\ &\int_{V'} \exp\{\frac{1}{t} < (\mu^{\mathcal{N}}(X) + \sqrt{-1}\mu^{\mathcal{N}}(Y))\cdot, \cdot > + \frac{1}{2t} < \mu^{\mathcal{N}}(X)y + \sqrt{-1}\mu^{\mathcal{N}}(Y)y, R^{\mathcal{N}}y >\}\eta \\ &+ \int_{V'} \exp\{-\frac{1}{2t} < -\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y, -\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y >\}\eta \end{aligned}$$

By making the change of variables $y = \sqrt{t}y$, we find that the above formula is equal to

$$t^n \int_{V'} \exp\{\frac{1}{t} < (\mu^{\mathcal{N}}(X) + \sqrt{-1}\mu^{\mathcal{N}}(Y))\cdot, \cdot > + \frac{1}{2} < \mu^{\mathcal{N}}(X)y + \sqrt{-1}\mu^{\mathcal{N}}(Y)y, R^{\mathcal{N}}y >\}\eta$$

$$+ \int_{V'} \exp\{-\frac{1}{2} < -\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y, -\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y >\}\eta_{\sqrt{t}y}$$

we known that

$$\frac{(\frac{<(\mu^{\mathcal{N}}(X)+\sqrt{-1}\mu^{\mathcal{N}}(Y))\cdot,\cdot>}{t})^n}{n!} = (\mathrm{Pf}(\mu^{\mathcal{N}}(\mathrm{X}) + \sqrt{-1}\mu^{\mathcal{N}}(\mathrm{Y})))\mathrm{dy}$$

here dy is the volume form of the submanifold M_0, let 2n be the dimension of M_0, then we get

$$= \int_{V'} \exp\{\frac{1}{2} < \mu^{\mathcal{N}}(X)y + \sqrt{-1}\mu^{\mathcal{N}}(Y)y, R^{\mathcal{N}}y >\}\eta \det(\mu^{\mathcal{N}}(X) + \sqrt{-1}\mu^{\mathcal{N}}(Y))^{\frac{1}{2}} dy_1 \wedge ... \wedge dy_n$$

$$+ \int_{V'} \exp\{-\frac{1}{2} < -\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y, -\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y >\}\eta$$

Because by $[X_M, Y_M] = 0$ we have $[\mu^{TM}(X), \mu^{TM}(Y)] = 0$. And by $-\mu^{\mathcal{N}}(X) - \sqrt{-1}\mu^{\mathcal{N}}(Y)$, $R^{\mathcal{N}}$ are skew-symmetric, so we get

$$= \int_{V'} \exp\{-\frac{1}{2} < -\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y, -\mu^{\mathcal{N}}(X)y - \sqrt{-1}\mu^{\mathcal{N}}(Y)y + R^{\mathcal{N}}y >\} dy_1 \wedge ... \wedge dy_n$$

$$\cdot \det(\mu^{\mathcal{N}}(X) + \sqrt{-1}\mu^{\mathcal{N}}(Y))^{\frac{1}{2}}\eta$$

$$= \int_{M_0} (2\pi)^n \det(\mu^{\mathcal{N}}(X) + \sqrt{-1}\mu^{\mathcal{N}}(Y))^{-\frac{1}{2}} \det(-\mu^{\mathcal{N}}(X) - \sqrt{-1}\mu^{\mathcal{N}}(Y) + R^{\mathcal{N}})^{-\frac{1}{2}}$$

$$\cdot \det(\mu^{\mathcal{N}}(X) + \sqrt{-1}\mu^{\mathcal{N}}(Y))^{\frac{1}{2}}\eta$$

$$= \int_{M_0} (2\pi)^n \det(-\mu^{\mathcal{N}}(X) - \sqrt{-1}\mu^{\mathcal{N}}(Y) + R^{\mathcal{N}})^{-\frac{1}{2}}\eta$$

$$= \int_{M_0} \frac{\eta}{\mathrm{Pf}[\frac{-\mu^{\mathcal{N}}(\mathrm{X}) - \sqrt{-1}\mu^{\mathcal{N}}(\mathrm{Y}) + \mathrm{R}^{\mathcal{N}}}{2\pi}]}$$

□

By theorem 1.,we can get the localization formulas of Berline and Vergne(see [2] or [3]).

Corollery 1 (N.Berline and M.Vergne)**.** *Let M be a smooth closed oriented manifold, G be a compact Lie group acting smoothly on M. For any $\eta \in H_X^*(M)$, let G_0 be the Lie subgroup of G which preserves the submanifold $M_0 = \{x \in M \mid X_M(x) = 0\}$, the following identity hold:*

$$\int_M \eta = \int_{M_0} \frac{\eta}{\mathrm{Pf}[\frac{-\mu^{\mathcal{N}}(\mathrm{X}) + \mathrm{R}^{\mathcal{N}}}{2\pi}]}$$

Proof. Because $M_0 = \{x \in M \mid X_M(x) = 0\}$, we have $\exp(-tX)p = p$ for $p \in M_0$, so $\exp(-tX) \in G_0$. By theorem 1., we set $Y = 0$, then we get the result. □

4 Localization formulas for characteristic numbers

Let M be an even dimensional compact oriented manifold without boundary, G be a compact Lie group acting smoothly on M and $\mathfrak{g}$ be its Lie algebra. Let g^{TM} be a G-invariant Riemannian metric on TM, ∇^{TM} is the Levi-Civita connection associated to g^{TM}. Here ∇^{TM} is a G-invariant connection, we see that $[\nabla^{TM}, L_{X_M}] = 0$ for all $X \in \mathfrak{g}$.

The equivariant connection $\widetilde{\nabla}^{TM}$ is the operator on $\Omega^*(M, TM)$ corresponding to a G-invariant connection ∇^{TM} is defined by the formula

$$\widetilde{\nabla}^{TM} = \nabla^{TM} + i_{X_M + \sqrt{-1}Y_M}$$

here X_M, Y_M be the smooth vector field on M corresponded to $X, Y \in \mathfrak{g}$.

Lemma 5. *The operator $\widetilde{\nabla}^{TM}$ preserves the space $\Omega^*_{X_M+\sqrt{-1}Y_M}(M, TM)$ which is the space of smooth $(X_M + \sqrt{-1}Y_M)$-invariant forms with values in TM.*

Proof. Let $\omega \in \Omega^*_{X_M+\sqrt{-1}Y_M}(M)$, then we have

$$\begin{aligned}(L_{X_M} + \sqrt{-1}L_{Y_M})\widetilde{\nabla}^{TM}\omega &= (L_{X_M} + \sqrt{-1}L_{Y_M})(\nabla^{TM} + i_{X_M+\sqrt{-1}Y_M})\omega \\ &= (\nabla^{TM} + i_{X_M+\sqrt{-1}Y_M})(L_{X_M} + \sqrt{-1}L_{Y_M})\omega \\ &= 0\end{aligned}$$

So we get $\widetilde{\nabla}^{TM}\omega \in \Omega^*_{X_M+\sqrt{-1}Y_M}(M, TM)$. □

We will also denote the restriction of $\widetilde{\nabla}^{TM}$ to $\Omega^*_{X_M+\sqrt{-1}Y_M}(M, TM)$ by $\widetilde{\nabla}^{TM}$.

The equivariant curvature $\widetilde{R}^{TM}$ of the equivariant connection $\widetilde{\nabla}^{TM}$ is defined by the formula(see [1])

$$\widetilde{R}^{TM} = (\widetilde{\nabla}^{TM})^2 - L_{X_M} - \sqrt{-1}L_{Y_M}$$

It is the element of $\Omega^*_{X_M+\sqrt{-1}Y_M}(M, End(TM))$. We see that

$$\begin{aligned}\widetilde{R}^{TM} &= (\nabla^{TM} + i_{X_M+\sqrt{-1}Y_M})^2 - L_{X_M} - \sqrt{-1}L_{Y_M} \\ &= R^{TM} + [\nabla^{TM}, i_{X_M+\sqrt{-1}Y_M}] - L_{X_M} - \sqrt{-1}L_{Y_M} \\ &= R^{TM} - \mu^{TM}(X) - \sqrt{-1}\mu^{TM}(Y)\end{aligned}$$

Lemma 6. *The equivariant curvature $\widetilde{R}^{TM}$ satisfies the equvariant Bianchi formula*

$$\widetilde{\nabla}^{TM}\widetilde{R}^{TM} = 0$$

Proof. Because

$$\begin{aligned}[\widetilde{\nabla}^{TM}, \widetilde{R}^{TM}] &= [\widetilde{\nabla}^{TM}, (\widetilde{\nabla}^{TM})^2 - L_{X_M} - \sqrt{-1}L_{Y_M}] \\ &= [\widetilde{\nabla}^{TM}, (\widetilde{\nabla}^{TM})^2] + [\nabla^{TM} + i_{X_M+\sqrt{-1}Y_M}, -L_{X_M} - \sqrt{-1}L_{Y_M}] \\ &= 0\end{aligned}$$

□

Now we to construct the equivariant characteristic forms by $\widetilde{R}^{TM}$. If $f(x)$ is a polynomial in the indeterminate x, then $f(\widetilde{R}^{TM})$ is an element of $\Omega^*_{X_M+\sqrt{-1}Y_M}(M, End(TM))$. We use the trace map

$$\mathrm{Tr} : \Omega^*_{X_M+\sqrt{-1}Y_M}(M, End(TM)) \to \Omega^*_{X_M+\sqrt{-1}Y_M}(M)$$

to obtain an element of $\Omega^*_{X_M+\sqrt{-1}Y_M}(M)$, which we call an equivariant characteristic form.

Lemma 7. *The equivariant differential form* $\mathrm{Tr}(f(\widetilde{R}^{TM}))$ *is* $d_{X_M+\sqrt{-1}Y_M}$*-closed, and its equivariant cohomology class is independent of the choice of the G-invariant connection* ∇^{TM}*.*

Proof. If $\alpha \in \Omega^*_{X_M+\sqrt{-1}Y_M}(M, End(TM))$, because in local $\nabla^{TM} = d + \omega$, we have

$$\begin{aligned} d_{X_M+\sqrt{-1}Y_M}\mathrm{Tr}(\alpha) &= \mathrm{Tr}(d_{X_M+\sqrt{-1}Y_M}\alpha) \\ &= \mathrm{Tr}([d_{X_M+\sqrt{-1}Y_M}, \alpha]) + \mathrm{Tr}([\omega, \alpha]) \\ &= \mathrm{Tr}([\widetilde{\nabla}^{TM}, \alpha]) \end{aligned}$$

then by the equivariant Bianchi identity $\widetilde{\nabla}^{TM}\widetilde{R}^{TM} = 0$, we get

$$d_{X_M+\sqrt{-1}Y_M}\mathrm{Tr}(f(\widetilde{R}^{TM})) = 0.$$

Let ∇_t^{TM} is a one-parameter family of G-invariant connections with equivariant curvature $\widetilde{R}_t^{TM}$. We have

$$\begin{aligned} \frac{d}{dt}\mathrm{Tr}(f(\widetilde{R}_t^{TM})) &= \mathrm{Tr}(\frac{d\widetilde{R}_t^{TM}}{dt}f'(\widetilde{R}_t^{TM})) \\ &= \mathrm{Tr}(\frac{d(\widetilde{\nabla}_t^{TM})^2}{dt}f'(\widetilde{R}_t^{TM})) \\ &= \mathrm{Tr}([\widetilde{\nabla}_t^{TM}, \frac{d\widetilde{\nabla}_t^{TM}}{dt}]f'(\widetilde{R}_t^{TM})) \\ &= \mathrm{Tr}([\widetilde{\nabla}_t^{TM}, \frac{d\widetilde{\nabla}_t^{TM}}{dt}f'(\widetilde{R}_t^{TM})]) \\ &= d_{X_M+\sqrt{-1}Y_M}\mathrm{Tr}(\frac{d\widetilde{\nabla}_t^{TM}}{dt}f'(\widetilde{R}_t^{TM})) \end{aligned}$$

from which we get

$$\mathrm{Tr}(f(\widetilde{R}_1^{TM})) - \mathrm{Tr}(f(\widetilde{R}_0^{TM})) = d_{X_M+\sqrt{-1}Y_M}\int_0^1 \mathrm{Tr}(\frac{d\widetilde{\nabla}_t^{TM}}{dt}f'(\widetilde{R}_t^{TM}))dt$$

so we get the result. □

As an application of Theorem 1., we can get the following localization formulas for characteristic numbers

Theorem 2. *Let* M *be an* $2l$*-dim compact oriented manifold without boundary,* G *be a compact Lie group acting smoothly on* M *and* $\mathfrak{g}$ *be its Lie algebra. Let* $X, Y \in \mathfrak{g}$*, and* X_M, Y_M *be the corresponding smooth vector field on* M*.* M_0 *is the submanifold descriped in section 2. If* $f(x)$ *is a polynomial, then we have*

$$\int_M \mathrm{Tr}(f(\widetilde{R}^{TM})) = \int_{M_0} \frac{\mathrm{Tr}(f(\widetilde{R}^{TM}))}{\mathrm{Pf}[\frac{-\mu^{\mathcal{N}}(\mathrm{X})-\sqrt{-1}\mu^{\mathcal{N}}(\mathrm{Y})+\mathrm{R}^{\mathcal{N}}}{2\pi}]}$$

Proof. By Lemma 7., we have $\mathrm{Tr}(f(\widetilde{R}^{TM}))$ is $d_{X_M+\sqrt{-1}Y_M}$-closed. And by Theorem 1., we get the result. □

We can use this formula to compute these characteristic numbers of M, especially we can use it to Euler characteristic of M. Here we didn't to give the details.

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